

TORIC DEGENERATIONS OF SCHUBERT VARIETIES

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ABSTRACT. *Let G be a simply connected semi-simple complex algebraic group. We prove that every Schubert variety of G has a flat degeneration into a toric variety. This provides a generalization of results of [7], [6], [5]. Our basic tool is Lusztig's canonical basis and the string parametrization of this basis.*

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0. Introduction.

0.1. Let G be a simply connected semi-simple complex algebraic group. Fix a maximal torus T and a Borel subgroup B such that $T \subset B \subset G$. Let W the Weyl group of G relative to T . For any w in W , let $X_w = \overline{BwB/B}$ denote the Schubert variety corresponding to w . This article is concerned with the following problem.

Degeneration Problem. *Is there a flat family over $\text{Spec}\mathbb{C}[t]$, such that the general fiber is X_w and the special fiber is a toric variety?*

The existence of such a degeneration was obtained by N. Gonciulea and V. Lakshmibai for $G = SL_n$, [7]. Their proof is based on the theory of standard monomials. In the case $G = SL_n$, the corner stone of their proof is the following : fundamental weights are minuscule weights, hence, a basis of every fundamental representation is endowed with a structure of distributive lattice.

A toric degeneration for Schubert varieties is given in [5], [6], for G of rank 2. The proofs rely on the theory of standard monomials as well. A natural question would be : is there a (flat) toric degeneration of the flag variety G/B which restricts to a toric

degeneration of the Schubert varieties X_w for any w in the Weyl group? In [4], R. Chirivì gives a degeneration of the flag variety which restricts into semi-toric degenerations of the Schubert varieties, i.e. finite unions of irreducible toric varieties. An explanation of this fact was given to us by O. Mathieu : intersections of irreducible toric varieties are irreducible toric varieties, but intersection of Schubert varieties can be a union of several Schubert varieties. Hence, the answer to the previous question is negative. In [4], the Degeneration Problem is solved with toric replaced by semi-toric.

0.2. Our approach of the problem is based on the canonical/global base of Lusztig/Kashiwara and the so-called string parametrization of this base studied by P. Littelmann in [10] and made precise by A. Berenstein and A. Zelevinsky in [1].

Fix w in W . Let P^+ be the semigroup of dominant weights. For all λ in P^+ , let \mathcal{L}_λ be the line bundle on G/B corresponding to λ . Then, the direct sum of global sections $R_w := \bigoplus_{\lambda \in P^+} H^0(X_w, \mathcal{L}_\lambda)$ carries a natural structure of P^+ -graded \mathbb{C} -algebra. Moreover, there exists a natural action of T on R_w . Our principal result can be stated as follows :

Theorem. *Fix w in W . There exists a filtration $(\mathcal{F}_m^w)_{m \in \mathbb{N}}$ of R_w such that*

- (i) *for all m in \mathbb{N} , \mathcal{F}_m^w is compatible with the P^+ -grading of R_w ,*
- (ii) *for all m in \mathbb{N} , \mathcal{F}_m^w is compatible with the action of T ,*
- (iii) *the associated graded algebra is the \mathbb{C} -algebra of the semigroup of integral points in a rational convex polyhedral cone.*

This cone depends on the choice of a reduced decomposition \tilde{w}_0 of the longest element w_0 of the Weyl group. Explicit equations for the faces of this cone can be obtained from [10] for so-called nice decompositions \tilde{w}_0 . More generally, those equations were obtained in [1] from \tilde{w}_0 -trails in fundamental Weyl modules of the Langlands dual of G .

This theorem gives a positive answer to the Degeneration Problem. Indeed, let λ be a regular dominant weight, then the line bundle \mathcal{L}_λ is ample and X_w is the projective spectrum of $\bigoplus_{m \in \mathbb{N}} H^0(X_w, \mathcal{L}_{m\lambda})$. Moreover, the spectrum of a noetherian graded algebra associated to a filtration of a noetherian algebra R is a flat degeneration of $\text{Spec}(R)$. This is proved as follows by a standard argument : let t be an indeterminate and consider the filtration $(R_n)_{n \in \mathbb{N}}$ of R . Then, the $\mathbb{C}[t]$ -algebra $R^t = \bigoplus_n R_n t^n$ is flat over $\mathbb{C}[t]$ and it verifies $R^t/(t - t_0)R^t \simeq R$ for $t_0 \neq 0$ and $R^t/tR^t \simeq \text{Gr}R$.

Let U be the maximal unipotent subgroup of B . Hence, R_{w_0} is the algebra $\mathbb{C}[G/U]$ of regular functions on G/U . For any w , the algebras R_w are quotients of this algebra. If \tilde{w}_0 is adapted to w in the sense of Definition 2.4, then the filtration $(\mathcal{F}_m^w)_{m \in \mathbb{N}}$ is the quotient filtration of $(\mathcal{F}_m^{w_0})_{m \in \mathbb{N}}$. In general, the quotient filtration $(\overline{\mathcal{F}}_m^{w_0})_{m \in \mathbb{N}}$ of R_w provides a graded associated algebra whose spectrum is a semi-toric variety.

The proof of the theorem is based on two facts.

Let U^- be the maximal unipotent subgroup of G which is opposite to U . Set $B^- = TU^-$. Then, the algebra $\mathbb{C}[G/U]$ embeds in $\mathbb{C}[B^-]$. Moreover, we can embed the (specialized) dual of the canonical base in the algebra $\mathbb{C}[U^-]$. We prove that this dual has good multiplicative properties inherited from the quantum case, see Theorem 2.3. This part of the article is inspired by [13]. But here, we don't use the positivity arguments or the elaborate Hall algebra model, only true for the simply laced case.

In a second step we show how to restrict from flag variety to Schubert varieties : this part relies on the compatibility of the canonical base with the Demazure modules, [9], [10].

1. Notations and recollection on global basis.

1.1. Denote by G a semisimple simply connected complex Lie group. Fix a torus T of G and let B be a Borel subgroup such that $T \subset B \subset G$. Denote by U the unipotent radical of B . Let B^- be the opposite Borel subgroup and U^- be its unipotent radical. Let \mathfrak{g} , resp. \mathfrak{h} , \mathfrak{n} , \mathfrak{b} , \mathfrak{n}^- , \mathfrak{b}^- , be Lie \mathbb{C} -algebra of G , resp. T , U , B , U^- , B^- . Let n be the rank of \mathfrak{g} . We have the triangular decomposition $\mathfrak{g} = \mathfrak{n}^- + \mathfrak{h} + \mathfrak{n}$. Let $\{\alpha_i\}_i$ be a basis of the root system Δ corresponding to this decomposition. Let P be the weight lattice generated by the fundamental weights ϖ_i , $1 \leq i \leq n$, and let $P^+ := \sum_i \mathbb{N}\varpi_i$ be the semigroup of integral dominant weights. Let W be the Weyl group, generated by the reflections s_{α_i} corresponding to the simple roots α_i , and let w_0 be the longest element of W . We denote by (\cdot, \cdot) the W -invariant form on P .

1.2. Let d be an integer such that $(P, P) \subset (2/d)\mathbb{Z}$. Let q be a indeterminate and set $\mathbb{K} = \mathbb{C}(q^{1/d})$. Let $U_q(\mathfrak{g})$ be the simply connected quantized enveloping algebra on \mathbb{K} , as defined in [3]. Set $d_i = (\alpha_i, \alpha_i)/2$ and $q_i = q^{d_i}$ for all i . Let $U_q(\mathfrak{n})$, resp. $U_q(\mathfrak{n}^-)$, be the subalgebra generated by the canonical generators E_{α_i} , resp. F_{α_i} , of positive, resp. negative, weights and the quantum Serre relations. For all λ in P , let K_λ the corresponding element in the algebra $U_q^0 = \mathbb{K}[P]$ of the torus of $U_q(\mathfrak{g})$. We have the triangular decomposition $U_q(\mathfrak{g}) = U_q(\mathfrak{n}^-) \otimes U_q^0 \otimes U_q(\mathfrak{n})$. We set

$$U_q(\mathfrak{b}) = U_q(\mathfrak{n}) \otimes U_q^0, \quad U_q(\mathfrak{b}^-) = U_q(\mathfrak{n}^-) \otimes U_q^0.$$

$U_q(\mathfrak{g})$ is endowed with a structure of Hopf algebra and the comultiplication Δ , the antipode S and the augmentation ε are given by

$$\begin{aligned} \Delta E_i &= E_i \otimes 1 + K_{\alpha_i} \otimes E_i, \quad \Delta F_i = F_i \otimes K_{\alpha_i}^{-1} + 1 \otimes F_i, \quad \Delta K_\lambda = K_\lambda \otimes K_\lambda \\ S(E_i) &= -K_{\alpha_i}^{-2} E_i, \quad S(F_i) = -F_i K_{\alpha_i}^2, \quad S(K_\lambda) = K_{-\lambda} \\ \varepsilon(E_i) &= \varepsilon(F_i) = 0, \quad \varepsilon(K_\lambda) = 1. \end{aligned}$$

If M is a U_q^0 -module and $\gamma \in P$, we set $M_\gamma := \{m \in M, K_\lambda.m = q^{(\lambda, \gamma)}\}$.

For n a non negative integer and α a positive root, we set : $[n]_i = \frac{1-q_i^n}{1-q_i}$, $[n]_i! = [n]_i[n-1]_i \dots [1]_i$.

1.3. The dual $U_q(\mathfrak{g})^*$ is endowed with a structure of left, resp. right, $U_q(\mathfrak{g})$ -module by $u.c(a) = c(au)$, resp. $c.u(a) = c(ua)$, $u, a \in U_q(\mathfrak{g})$, $c \in U_q(\mathfrak{g})^*$. If M is a finite dimensional left $U_q(\mathfrak{g})$ -module, we endow the dual M^* with a structure of left $U_q(\mathfrak{g})$ -module by $u\xi(v) = \xi(S(u)v)$, $u \in U_q(\mathfrak{g})$, $\xi \in M^*$, $v \in M$.

For all λ in P^+ , let $V_q(\lambda)$ be the simple $U_q(\mathfrak{g})$ -module with highest weight λ . We can embed $V_q(\lambda)^* \otimes V_q(\lambda)$ in $U_q(\mathfrak{g})^*$ by setting $\xi \otimes v(u) = \xi(u.v)$, $u \in U_q(\mathfrak{g})$, $\xi \in V_q(\lambda)^*$,

$v \in V_q(\lambda)$. Let v_λ be a highest weight vector of $V_q(\lambda)$. For all integral dominant weight λ , let $C(\lambda)$, resp. $C^+(\lambda)$, be the subspace of $U_q(\mathfrak{g})^*$ generated by the $\xi \otimes v$, resp. $\xi \otimes v_\lambda$, $\xi \in V_q(\lambda)^*$, $v \in V_q(\lambda)$. We set $\mathbb{C}_q[G] = \bigoplus_{\lambda \in P^+} C(\lambda)$, $\mathbb{C}_q[G/U] = \bigoplus_{\lambda \in P^+} C^+(\lambda)$. Then, $\mathbb{C}_q[G]$ and $\mathbb{C}_q[G/U]$ are subalgebras of the Hopf dual of $U_q(\mathfrak{g})$. $\mathbb{C}_q[G]$, resp. $\mathbb{C}_q[G/U]$, is the algebra of quantum regular functions on G , resp. on the quotient G/U .

1.4. There exists a unique bilinear form (\cdot, \cdot) on $U_q(\mathfrak{b}) \times U_q(\mathfrak{b}^-)$, see [14], [15], [3], such that :

$$(1.4.1) \quad (u^+, u_1^- u_2^-) = (\Delta(u^+), u_1^- \otimes u_2^-), \quad u^+ \in U_q(\mathfrak{b}); u_1^-, u_2^- \in U_q(\mathfrak{b}^-)$$

$$(1.4.2) \quad (u_1^+ u_2^+, u^-) = (u_2^+ \otimes u_1^+, \Delta(u^-)), \quad u^- \in U_q(\mathfrak{b}^-); u_1^+, u_2^+ \in U_q(\mathfrak{b})$$

$$(1.4.3) \quad (K_\lambda, K_\mu) = q^{-(\lambda, \mu)}, \quad \lambda, \mu \in P$$

$$(1.4.4) \quad (K_\lambda, F_i) = 0, \quad \lambda \in P, 1 \leq i \leq n$$

$$(1.4.5) \quad (E_i, K_\lambda) = 0, \quad \lambda \in P, 1 \leq i \leq n$$

$$(1.4.6) \quad (E_i, F_i) = \delta_{ij} (1 - q_i^2)^{-1}, \quad 1 \leq i, j \leq n.$$

For all β in Q^+ , let $U_q(\mathfrak{n})_\beta$, resp. $U_q(\mathfrak{n}^-)_{-\beta}$, be the subspace of $U_q(\mathfrak{n})$, resp. $U_q(\mathfrak{n}^-)$, with weight β , resp. $-\beta$. The form (\cdot, \cdot) is non degenerate on $U_q(\mathfrak{n})_\beta \times U_q(\mathfrak{n}^-)_{-\beta}$, $\beta \in Q^+$. We have, by (1.4.1-1.4.5) :

$$(1.4.7) \quad (X K_\lambda, Y K_\mu) = q^{-(\lambda, \mu)} (X, Y), \quad X \in U_q(\mathfrak{n}), Y \in U_q(\mathfrak{n}^-)$$

We can define a bilinear form $\langle \cdot, \cdot \rangle$ on $U_q(\mathfrak{g}) \times U_q(\mathfrak{g})$ by :

$$(1.4.8) \quad \langle X_1 K_\lambda S(Y_1), Y_2 K_\mu S(X_2) \rangle = (X_1, Y_2) (X_2, Y_1) q^{-(\lambda, \mu)/2}$$

where $X_1, X_2 \in U_q(\mathfrak{n})$, $Y_1, Y_2 \in U_q(\mathfrak{n}^-)$, $\lambda, \mu \in P$. This form is non degenerate.

There exists an algebra isomorphism from $U_q(\mathfrak{n})$ to $U_q(\mathfrak{n}^-)$ which maps E_i on F_i for all i . Via this isomorphism, the restriction of the form (\cdot, \cdot) on $U_q(\mathfrak{n}) \times U_q(\mathfrak{n}^-)$ coincides with the one defined by Lusztig in [11, par 1].

1.5. Define the maps :

$$\beta : U_q(\mathfrak{b}) \rightarrow U_q(\mathfrak{b}^-)^*, \quad \beta(u)(v) = (u, v);$$

$$\zeta : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})^*, \quad \zeta(u)(v) = \langle u, v \rangle.$$

It follows from 1.4 that :

Lemma. *With the previous notations, we have*

- (i) β, ζ are injective.
- (ii) β is an anti-homomorphism of algebras.

Denote by ρ the restriction homomorphism from $U_q(\mathfrak{g})^*$ onto $U_q(\mathfrak{b}^-)^*$. We know, see [2, Proposition 3.4],

Proposition. *The restriction of ρ to $\mathbb{C}_q[G/U]$ is injective. Moreover, for all λ in P^+ , we have*

- (i) *For all e in $U_q(\mathfrak{n})$, $\rho(\zeta(eK_{-2\lambda})) = \beta(eK_{-\lambda})$.*
- (ii) *there exists a (unique) subspace E_λ of $U_q(\mathfrak{n})$ such that $\zeta(E_\lambda K_{-2\lambda}) = C^+(\lambda)$. \diamond*

1.6. Let $u \mapsto \bar{u}$ be the \mathbb{K} -antihomomorphism of $U_q(\mathfrak{g})$ such that $\bar{E}_i = E_i$, $\bar{K}_\lambda = K_{-\lambda}$, $\bar{F}_i = F_i$. It is easily seen that

$$\langle u, v \rangle = (u, v) = \langle \bar{u}, \bar{v} \rangle, \quad u \in U_q(\mathfrak{n}), v \in U_q(\mathfrak{n}^-).$$

Let \mathcal{B} be Lusztig's canonical basis of $U_q(\mathfrak{n}^-)$, [11], which coincides with Kashiwara's global basis, [9]. Let $\mathcal{B}^* \subset U_q(\mathfrak{n})$ be the dual basis in $U_q(\mathfrak{n})$, i.e. $(b^*, b') = \delta_{b, b'}$. Let \tilde{E}_i , $\tilde{F}_i : U_q(\mathfrak{n}^-) \rightarrow U_q(\mathfrak{n}^-)$ be the Kashiwara operators, [loc. cit.]. For $b \in \mathcal{B}$, $\tilde{E}_i(b)$, resp. $\tilde{F}_i(b)$, equals some b' in $\mathcal{B} \cup \{0\}$ modulo $q^{-1}\mathbb{Z}[q^{-1}]\mathcal{B}$. The assignment $b \mapsto b'$ defines maps \tilde{e}_i and \tilde{f}_i from \mathcal{B} to $\mathcal{B} \cup \{0\}$. For $b \in \mathcal{B}$, $1 \leq i \leq n$, set $\varepsilon_i(b) = \text{Max}\{r, \tilde{e}_i^r(b) \neq 0\}$.

Let L_i , $1 \leq i \leq n$, be the adjoint of the left multiplication operator F_{i-} for the form (\cdot, \cdot) on $U_q(\mathfrak{n}) \times U_q(\mathfrak{n}^-)$. Then, L_i is a quantum derivation of $U_q(\mathfrak{n})$, [11, par 1] :

$$L_i(e_\alpha u) = L_i(e_\alpha)u + q^{(\alpha, \alpha_i)}e_\alpha L_i(u), \quad u \in U_q(\mathfrak{n}), e_\alpha \in U_q(\mathfrak{n})_\alpha, \alpha \in Q.$$

Set $L_i^{(r)} = \frac{1}{[r]_i!} L_i^r$.

The following is a recollection of results about the canonical basis and its dual. Assertions (i) and (ii) can be read in [11, 14.4.13, 14.4.14]. Assertion (iii) is a standard consequence of [11, 14.3.2 (c)] by dualization.

Theorem 1. *For b, b' in \mathcal{B} , we have :*

- (i) $bb' \in \mathbb{Z}[q, q^{-1}]\mathcal{B}$,
- (ii) $b^*b'^* \in \mathbb{Z}[q, q^{-1}]\mathcal{B}^*$,
- (iii) $L_i^{(\varepsilon_i(b))}(b^*) = (\tilde{e}_i^{\varepsilon_i(b)}(b))^*$,
- (iv) $\bar{b} \in \mathcal{B}$. \diamond

The following theorem states precisely the compatibility of the canonical basis with the finite dimensional highest weight modules. It can be found in [9, Proposition 8.2].

Theorem 2. *Fix λ in P^+ , $\lambda = \sum_i \lambda_i \varpi_i$. Set*

$$\mathcal{B}_\lambda := \{b \in \mathcal{B}, \varepsilon_i(\bar{b}) \leq \lambda_i, 1 \leq i \leq n\}.$$

Then $\mathcal{B}_\lambda = \{b \in \mathcal{B}, b.v_\lambda \neq 0\}$ and $\mathcal{B}_\lambda.v_\lambda$ is a basis of $V_q(\lambda)$. \diamond

When b belongs to \mathcal{B}_λ , and if no confusion occurs, we shall use the same symbol for b and $b.v_\lambda$, i.e. we set $b = b.v_\lambda$.

1.7. Let $A = \mathbb{C}[q, q^{-1}]$. Let $U_A(\mathfrak{n}^-)$ be the A -submodule of $U_q(\mathfrak{n}^-)$ generated by \mathcal{B} . Then, $U_A(\mathfrak{n}^-)$ is a free A -space and a A -algebra. Indeed, $U_A(\mathfrak{n}^-)$ is the A -algebra generated by the $\frac{1}{[m]_i!} F_i^m$, see [8, Theorem 11.10 (b)]. Let U_A be the sub- A -algebra of $U_q(\mathfrak{g})$ generated by the $\frac{1}{[m]_i!} F_i^m$, and the $\frac{1}{[m]_i!} E_i^m$. Then, $V_A(\lambda)$ is a U_A -module. Set $V_A(\lambda) = U_q(\mathfrak{n}^-).v_\lambda \subset V_q(\lambda)$. Then, $V_A(\lambda)$ is the A -space generated by \mathcal{B}_λ . By [8, Theorem 11.19], we know that \mathcal{B} is compatible with specialisation :

$$\mathbb{C} \otimes_A U_A(\mathfrak{n}^-) \simeq U(\mathfrak{n}^-), \quad \mathbb{C} \otimes_A V_A(\lambda) \simeq V(\lambda), \quad \lambda \in P^+,$$

where $\mathbb{C} = A/(q-1)A$ as a A -module, $U(\mathfrak{n}^-)$ is the (classical) enveloping \mathbb{C} -algebra of \mathfrak{n}^- and $V(\lambda)$ is the classical Weyl module with highest weight λ .

Let $V_A(\lambda)^*$ be the A -dual of $V_A(\lambda)$. Then, it has a natural U_A -module structure by $u.v^*(_) = v^*(S(u)._)$. The module $V_A(\lambda)^*$ specializes at $q = 1$ onto the dual \mathfrak{g} -module $V(\lambda)^*$.

1.8. Fix λ in P^+ and w in W . We know that $V_q(\lambda)$ verifies the Weyl character formula ; we denote by $v_{w\lambda}$ an extremal vector of weight $w\lambda$. Then, the $U_q(\mathfrak{b})$ -module $V_{q,w}(\lambda) := U_q(\mathfrak{n})v_{w\lambda}$ verifies the Demazure character formula. We know, [9, Theorem 12.4], [12, 5.3-5.4], that

Theorem. *There exists a subset \mathcal{B}_w of \mathcal{B} such that $V_{q,w}(\lambda)$ is spanned by $\mathcal{B}_w.v_\lambda$. Moreover, if b is in \mathcal{B}_w then $\Delta(b) \in \langle \mathcal{B}_w \rangle \otimes \langle K_\mu \mathcal{B}_w, \mu \in P \rangle$.* \diamond

In particular, the orthogonal $V_{q,w}(\lambda)^\perp$ in $V_q(\lambda)$ of the Demazure module $V_{q,w}(\lambda)$ is generated as a space by $\mathcal{B}(\lambda) \cap \mathcal{B} \setminus \mathcal{B}_w$ and the dual $V_{q,w}(\lambda)^*$ is generated by the image of $\mathcal{B}(\lambda) \cap \mathcal{B}_w$ by the quotient morphism.

As in 1.7, this allows us to define A -forms for Demazure modules. We denote by $V_{A,w}(\lambda)$ the A -space generated by $\mathcal{B}(\lambda) \cap \mathcal{B}_w$. It specializes for $q = 1$ to the classical Demazure module $V_w(\lambda)$.

2. A multiplicative property and Littelmann's parametrization of the dual canonical basis.

2.1. For λ in P^+ and b in \mathcal{B}_λ , let b_λ^* be the element of $V_q(\lambda)^*$ such that $b_\lambda^*(b'.v_\lambda) = \delta_{b,b'}$, $b' \in \mathcal{B}_\lambda$, where δ means the Kroenecker symbol.

Lemma. *For all λ in P^+ and b in \mathcal{B}_λ , then $\zeta(b^*K_{-2\lambda}) = b_\lambda^* \otimes v_\lambda$, $\beta(b^*K_{-\lambda}) = \rho(b_\lambda^* \otimes v_\lambda)$.*

Proof. By Lemma 1.5, we only need to prove that $\langle b^*K_{-2\lambda}, _ \rangle = b_\lambda^* \otimes v_\lambda$. As $U_q(\mathfrak{g}) = U_q(\mathfrak{b}^-) \oplus (U_q(\mathfrak{b}^-) \otimes U_q(\mathfrak{n}) \cap \text{Ker}(\varepsilon))$, we only need to prove this on $U_q(\mathfrak{b}^-)$, whose basis is given by $(b'K_\mu, b' \in \mathcal{B}, \mu \in P)$. By (1.4.7) and (1.4.8)

$$\langle b^*K_{-2\lambda}, b'K_\mu \rangle = (b^*, b')q^{\lambda, \mu} = \delta_{b,b'}q^{\lambda, \mu} = b_\lambda^*(b'K_\mu \cdot v_\lambda).$$

This implies the lemma. \diamond

Remark. By Proposition 1.5, the lemma implies that E_λ is spanned by \mathcal{B}_λ^* .

Let $A_q[G/U]$ be the sub- A -module of $\mathbb{C}_q[G]$ generated by the $b_\lambda^* \otimes v_\lambda$, $\lambda \in P^+$ and $b \in \mathcal{B}_\lambda$. Let $d_{b,b'}^{b''}$ be the coefficient of b''^* in the product $b^*b'^*$.

Proposition. *We have*

- (i) $\mathbb{K} \otimes_A A_q[G/U] = \mathbb{C}_q[G]$
- (ii) $A_q[G/U]$ is a A -algebra.
- (iii) if $d_{b',b}^{b''}$ is non zero, then $b \in \mathcal{B}_\lambda$, $b' \in \mathcal{B}_{\lambda'}$ implies $b'' \in \mathcal{B}_{\lambda+\lambda'}$,
- (iv) $A_q[G/U]/(q-1)A_q[G/U] \simeq \mathbb{C}[G/U]$, where $\mathbb{C}[G/U]$ is the \mathbb{C} -algebra of regular functions on G/U .

Proof. The A -basis b_λ^* of $V_A(\lambda)^*$ is a \mathbb{K} -basis of $V_q(\lambda)^*$. Hence, (i) is clear.

From Lemma 1.5, Proposition 1.5, and the previous lemma, we have

$$\begin{aligned} \beta^{-1}[(b_\lambda^* \otimes v_\lambda).(b'_{\lambda'}^* \otimes v_{\lambda'})] &= \beta^{-1}(b'_{\lambda'}^* \otimes v_{\lambda'}) \cdot \beta^{-1}(b_\lambda^* \otimes v_\lambda) \\ &= (b'^* K_{-\lambda'}).(b^* K_{-\lambda}) = q^{-(\lambda', \omega(b^*))} b'^* b^* K_{-\lambda-\lambda'} = q^{-(\lambda', \omega(b^*))} (\sum d_{b',b}^{b''} b''^*) K_{-\lambda-\lambda'}. \end{aligned}$$

We know that $(b_\lambda^* \otimes v_\lambda).(b'_{\lambda'}^* \otimes v_{\lambda'})$ belongs to $C^+(\lambda+\lambda')$ which is generated by $b^0 \otimes v_{\lambda+\lambda'}$, $b^0 \in \mathcal{B}_{\lambda+\lambda'}$. Hence, applying β^{-1} , we obtain (iii) and

$$(2.1.1) \quad (b_\lambda^* \otimes v_\lambda).(b'_{\lambda'}^* \otimes v_{\lambda'}) = q^{-(\lambda', \omega(b^*))} \sum d_{b',b}^{b''} (b''_{\lambda+\lambda'}^* \otimes v_{\lambda+\lambda'}).$$

This gives (ii). Now, (iv) is clear by 1.7 and the fact that specialization commutes with tensor product. \diamond

Corollary. *Fix b, b', b'' in \mathcal{B} with $d_{b,b'}^{b''}$ non zero. Then, for all i , $1 \leq i \leq n$, we have $\varepsilon_i(b'') \leq \varepsilon_i(b) + \varepsilon_i(b')$.*

Proof. By applying the antiautomorphism $\bar{}$, we obtain that $d_{\bar{b},\bar{b}}^{\bar{b}''} = d_{b,b'}^{b''}$. Set $\lambda = \sum_i \varepsilon_i(b) \varpi_i$ and $\lambda' = \sum_i \varepsilon_i(b') \varpi_i$. As $\bar{}$ is an involution, we deduce from [1.6, Theorem 2] that $\bar{b} \in \mathcal{B}_\lambda$ and $\bar{b}' \in \mathcal{B}_{\lambda'}$. This gives $\bar{b}'' \in \mathcal{B}_{\lambda+\lambda'}$ by (iii). Hence, we obtain the corollary from [1.6, Theorem 2]. \diamond

Remark. In the simply laced case, this corollary is easily obtained by the positivity property of the dual canonical basis, i.e. $d_{b,b'}^{b''} \in \mathbb{N}[q, q^{-1}]$ by [11]. For general \mathfrak{g} , we can conclude by a standard argument, transmitted by E. Vasserot. It is based on the realization of $d_{b,b'}^{b''}$ in terms of traces of an automorphism of a diagram on spaces arising from perverse sheaves, see [11, 14.4.14].

2.2. We introduce Littelmann's parametrization of the (dual) canonical basis. Fix a reduced decomposition of the longest element of the Weyl group $w_0 : \tilde{w}_0 = s_{i_1} \dots s_{i_N}$, where $N = \dim \mathfrak{n}$. For all u in $U_q(\mathfrak{n})$ and $1 \leq i \leq n$, set

$$a_i(u) = \text{Max}\{r, L_i^r(u) \neq 0\}, \quad \Lambda_i(u) = L_i^{(a_i(u))}(u).$$

For all b in \mathcal{B} , set

$$A_{\tilde{w}_0}(b) = (a_{i_N}(b^*), a_{i_{N-1}}(\Lambda_{i_N}(b^*)), \dots, a_{i_1}(\Lambda_{i_2} \dots \Lambda_{i_N}(b^*))) \in \mathbb{N}^N.$$

This parametrization can be found in [10, par 1]. It coincides with the parametrization in [1, 3.2] by [1.6, Theorem 1 (iii)]. We now present a theorem due to Littelmann, [10, par 1], see also [1, 3.10].

Theorem. *The map $A_{\tilde{w}_0}$ embeds \mathcal{B} into \mathbb{N}^N . Let \mathcal{C} be its image. Then, \mathcal{C} is the set of integral points of a rational convex polyhedral cone of \mathbb{R}^N . Moreover, set $\Gamma_{\tilde{w}_0} := \{(\lambda, \psi) \in P^+ \times \mathcal{C}, \psi \in A_{\tilde{w}_0}(\mathcal{B}_\lambda)\}$. Then, $\Gamma_{\tilde{w}_0}$ is the set of integral points of a rational convex polyhedral cone of \mathbb{R}^{n+N} . \diamond*

Note that equations of this cone can be given in [1, 3.10]. For all ψ in \mathcal{C} , set $b^\psi = b_{\tilde{w}_0}^\psi = A_{\tilde{w}_0}^{-1}(\psi)$. We also set $d_{\psi, \psi'}^{\psi''} := d_{b^\psi, b^{\psi'}}^{b''}$.

2.3. Let \prec be the lexicographical ordering of \mathbb{N}^N . We have

Theorem. *Fix a reduced decomposition \tilde{w}_0 of the longest element of the Weyl group. Let b, b', b'' be in \mathcal{B} , $A_{\tilde{w}_0}(b) = \psi$, $A_{\tilde{w}_0}(b') = \psi'$. Then, $d_{b, b'}^{b''}$ non zero implies $A_{\tilde{w}_0}(b'') \prec \psi + \psi'$. Moreover, $d_{\psi, \psi'}^{\psi + \psi'}$ is a power of q .*

Proof. First remark that $a_i(uv) = a_i(u) + a_i(v)$, u, v in $U_q(\mathfrak{n})$, by the quantized Leibniz rule. Write

$$(*) \quad b^* b'^* = \sum d_{b, b'}^{b''} b''^*$$

Let $\psi'' = A_{\tilde{w}_0}(b'')$, for b'' in the sum. Then, $\psi''_1 \leq \psi_1 + \psi'_1$ by Corollary 2.1 and [1.6, Theorem 1 (iii)]. This gives the first step of the induction. Now, applying $L_{i_N}^{\psi_1 + \psi'_1}$ and the Leibniz rule gives $\xi_1 \Lambda_{i_N}(b^*) \Lambda_{i_N}(b'^*) = \sum d_{b, b'}^{b''} \Lambda_{i_N}(b''^*)$, where the sum is taken over the elements b'' such that $\psi''_1 = \psi_1 + \psi'_1$, and where ξ_1 is a power of q . Now remark that, by [1.6, Theorem 1 (iii)], $\Lambda_{i_N}(b^*)$, $\Lambda_{i_N}(b'^*)$, and the $\Lambda_{i_N}(b''^*)$ all belong to the dual of the canonical basis. So, we can proceed as for the first step by induction. We then have the first assertion of the theorem. For conclusion, note that we obtain

$$\xi_N \Lambda_{i_1} \dots \Lambda_{i_N}(b^*) \Lambda_{i_1} \dots \Lambda_{i_N}(b'^*) = \sum d_{b, b'}^{b''} \Lambda_{i_1} \dots \Lambda_{i_N}(b''^*),$$

where the sum is taken over the elements b'' such that $\psi''_k = \psi_k + \psi'_k$, for all k and where ξ_N is a power of q . Hence, by Theorem 2.2 there is at most one element in the sum and for this element we have $A_{\tilde{w}_0}(b'') = \psi + \psi'$. By [10, par 1],

Lemma. $\Lambda_{i_1} \dots \Lambda_{i_N}(B^*) = 1$ for all elements B of the canonical basis. \diamond

Hence the coefficient $d_{\psi, \psi'}^{\psi + \psi'}$ is a power of q . This finishes the proof of the theorem. \diamond

2.4. By 1.8, the results of 2.1, 2.2 can be easily generalized to quotients of $\mathbb{C}_q[G/U]$ which correspond to Demazure modules. Indeed, fix w in W and let $\mathcal{B}_{\overline{w}}$ be the complement of \mathcal{B}_w in \mathcal{B} . Set

$$I_{A, w} := \bigoplus_{\lambda \in P^+, b \in \mathcal{B}_{\overline{w}} \cap \mathcal{B}(\lambda)} Ab^* \otimes v_\lambda = \bigoplus_{\lambda} V_{A, w}(\lambda)^\perp \otimes v_\lambda.$$

Then, $I_{A,w}$ is the orthogonal $\langle \mathcal{B}_w \rangle^\perp$ of $\langle \mathcal{B}_w \rangle$ in $A_q[G/U]$. By Theorem 1.8, $I_{A,w}$ is an ideal of $A_q[G/U]$. We have the following decomposition for the quotient algebra :

$$A_q[G/U]/I_{A,w} = \bigoplus_{\lambda} V_{A,w}(\lambda)^* \otimes v_{\lambda}.$$

From [10, par 1], we have :

Theorem. *Let $\tilde{w} = s_{i_1} \dots s_{i_p}$ be a reduced decomposition of w . Then there exists a reduced decomposition $\tilde{w}_0 = s_{i_N} \dots s_{i_1}$ of w_0 . For this decomposition, we have $A_{\tilde{w}_0}(B_w) = \mathcal{C} \cap (\mathbb{N}^p \times \{0\}^{N-p})$. \diamond*

Definition. For all w , a reduced decomposition $\tilde{w}_0 = s_{i_N} \dots s_{i_1}$ of w_0 such that $w = s_{i_1} \dots s_{i_p}$ will be called adapted to w .

3. Specialization.

3.1. Fix a reduced decomposition \tilde{w}_0 of w_0 . At this stage of the article, we can construct a \mathbb{N}^N -filtration of the algebra $\mathbb{C}[G/U]$ such that the associated graded algebra is the algebra of the semigroup $\Gamma_{\tilde{w}_0}$. To be more precise, let $b_{\lambda,\psi}^*$ in $\mathbb{C}[G/U]$ be the image of the element $(b_{\lambda}^{\psi})^* \otimes v_{\lambda}$ by the morphism of specialization at $q = 1$, see Proposition 2.1 (iv). We have by Theorem 2.3 and (2.1.1) :

Proposition. *The spaces $\mathcal{F}_{\psi} := \langle b_{\lambda,\phi}^*, (\lambda, \phi) \in \Gamma_{\tilde{w}_0}, \phi \prec \psi \rangle$, $\psi \in \mathcal{C}$ define a filtration of the algebra $\mathbb{C}[G/U]$. The graded associated algebra is naturally isomorphic to the \mathbb{C} -algebra of the semigroup $\Gamma_{\tilde{w}_0}$. \diamond*

3.2. What results from Proposition 3.1 is that there exists a finite sequence of degenerations of the flag variety which ends into a toric variety but what we want is a "degeneration in one step". Hence, we need a linear form $\mathbb{N}^N \rightarrow \mathbb{N}$, which satisfies some strict inequalities, and which transforms the \mathbb{N}^N -filtration of $\mathbb{C}[G/U]$ into a \mathbb{N} -filtration. This is made possible because the cone $\Gamma_{\tilde{w}_0}$ is convex polyhedral and hence has a finite presentation. We start by a lemma.

Lemma. *Let $\{\psi^1, \dots, \psi^K\}$ be a finite set of points in \mathbb{N}^N , and for all k , $1 \leq k \leq K$, let $\phi^{k,l}$, $1 \leq l \leq K_k$, be a finite number of points in \mathbb{N}^N such that $\phi^{k,l} \prec \psi^k$ for all l . Then, there exists a linear form $e : \mathbb{N}^N \rightarrow \mathbb{N}$ such that $e(\phi^{k,l}) < e(\psi^k)$ for all k and all l .*

Proof. Let a_k , $1 \leq k \leq N$ be the linear form of \mathbb{Q}^N which maps an element of \mathbb{Q}^N to its k -th coordinate. Set $I_0 = \{(\phi^{k,l}, \psi^k), 1 \leq k \leq K, 1 \leq l \leq K_k\}$. For s , $1 \leq s \leq N-1$, set $I_s = \{(\phi^{k,l}, \psi^k) \in I_0, (\phi^{k,l})_m = (\psi^k)_m, 1 \leq m \leq s\}$. All these sets are finite.

As a first step of our induction, define the linear form $e_N = a_N$ and fix ε_N in \mathbb{Q}^+ such that $e_N(\phi^{k,l})\varepsilon_N < (\psi^k)_{N-1} - (\phi^{k,l})_{N-1}$, for all $(\phi^{k,l}, \psi^k)$ in $I_{N-2} \setminus I_{N-1}$. Define a linear form by $e_{N-1} = a_{N-1} + \varepsilon_N e_N$. By construction, we have $e_{N-1}(\phi^{k,l}) < e_{N-1}(\psi^k)$ for all $(\phi^{k,l}, \psi^k)$ in I_{N-2} . Fix now ε_{N-1} in \mathbb{Q}^+ such that $e_{N-1}(\phi^{k,l})\varepsilon_N < (\psi^k)_{N-2} - (\phi^{k,l})_{N-2}$, for all $(\phi^{k,l}, \psi^k)$ in $I_{N-3} \setminus I_{N-2}$. Define a linear form by $e_{N-2} = a_{N-2} + \varepsilon_{N-1} e_{N-1}$. By construction and induction we have $e_{N-2}(\phi^{k,l}) < e_{N-2}(\psi^k)$ for all

$(\phi^{k,l}, \psi^k)$ in I_{N-3} . By induction, we obtain a form e_1 which verifies $e_1(\phi^{k,l}) < e_1(\psi^k)$ for all $(\phi^{k,l}, \psi^k)$ in I_0 . We can multiply e_1 by a positive integer in order to obtain a \mathbb{N} -form which verifies the lemma. \diamond

Let (λ_i, ψ_i) , $1 \leq i \leq K$, be the minimal set of generators of $\Gamma_{\tilde{w}_0}$ and $(\sum_i n_i^k(\lambda_i, \psi_i) - \sum_i m_i^k(\lambda_i, \psi_i))$, $n_i^k, m_i^k \in \mathbb{N}$, be the (finite) minimal set of generators of the relations. Then, from Proposition 3.1, we obtain :

Proposition. *The (commutative) algebra $\mathbb{C}[G/U]$ is defined by generators b_{λ_i, ψ_i}^* , $1 \leq i \leq K$, and relations*

$$(3.2.1) \quad \prod b_{\lambda_i, \psi_i}^{*n_i^k} = \prod b_{\lambda_i, \psi_i}^{*m_i^k} + \sum \mathbb{C}b_{\mu, \phi}$$

for $\phi \prec \sum_i n_i^k(\lambda_i, \psi_i) = \sum_i m_i^k(\lambda_i, \psi_i)$.

Proof. The algebra $\mathbb{C}[G/U]$ has a natural P^+ -grading defined by $\mathbb{C}[G/U]_\lambda = V(\lambda)^* \otimes v_\lambda$. By construction, $\mathcal{F}_\psi = \bigoplus_\lambda \mathcal{F}_{\psi, \lambda}$, where $\mathcal{F}_{\psi, \lambda} = \mathcal{F}_\psi \cap \mathbb{C}[G/U]_\lambda$. Clearly, $\mathcal{F}_{\psi, \lambda}$ is finite dimensional. Hence, by induction, we obtain that generators of the \mathcal{C} -graded algebra of Proposition 3.1 lift to generators of the algebra $\mathbb{C}[G/U]$. Hence, we have the first part of the proposition.

Now, Proposition 3.1 implies the relations (3.2.1). Let $\mathbb{C}[X_i, 1 \leq i \leq K]$ be the polynomial algebra with K indeterminates. There exists a surjective morphism $\varphi : \mathbb{C}[X_i, 1 \leq i \leq K]/J \rightarrow \mathbb{C}[G/U]$, where J is the ideal generated by the relations resulting from (3.2.1). This morphism maps X_i to b_{λ_i, ψ_i}^* for all i . Endow $\mathbb{C}[X_i, 1 \leq i \leq K]/J$ with the quotient filtration $< \prod X_i^{m_i} + J, \sum m_i \psi_i \prec \psi >_{\psi \in \mathcal{C}}$. Then, the associated graded algebra is defined by generators $\text{Gr}X_i$, $1 \leq i \leq K$, and relations $\prod \text{Gr}X_i^{n_i^k} = \prod \text{Gr}X_i^{m_i^k}$. Now, endow $\mathbb{C}[G/U]$ with its \mathcal{C} -filtration, see 3.1. By Proposition 3.1, $\text{Gr}\varphi$ is an isomorphism from $\text{Gr}\mathbb{C}[X_i, 1 \leq i \leq K]/J$ onto $\text{Gr}\mathbb{C}[G/U]$. This implies that φ is an isomorphism. This finishes the proof of the proposition. \diamond

Let $\psi^k := \sum_i n_i^k \psi_i$, and for each k , let the $\phi^{k,l}$ be the elements ϕ occurring in (3.2.1). Fix a \mathbb{N} -form e as in Lemma 3.2. Then,

Corollary. *The spaces $\mathcal{F}_m(\mathbb{C}[G/U]) = < \prod b_{\lambda_i, \psi_i}^{s_i^k}, e(\sum_i s_i^k \psi_i) \leq m >$, $m \in \mathbb{N}$, define a filtration of $\mathbb{C}[G/U]$. The graded associated algebra is $\mathbb{C}[\Gamma_{\tilde{w}_0}]$. \diamond*

3.3. We now give some analogous results for Demazure modules. Fix w in W with length p . Set $\Gamma_{\tilde{w}_0}^w := A_{\tilde{w}_0}(\mathcal{B}_w)$. Then,

Lemma. *For all reduced decompositions \tilde{w}_0 of w_0 , the set $\Gamma_{\tilde{w}_0}^w$ is a finite union of rational convex polyhedral cones. Moreover, if \tilde{w}_0 is adapted to w , see Definition 2.4, then $\Gamma_{\tilde{w}_0}^w$ is a p -dimensional face of $\Gamma_{\tilde{w}_0}$. In particular, it is a rational convex polyhedral cone.*

Proof. Fix w in W and fix a reduced decomposition \tilde{w}_0 adapted to w . By Theorem 2.4, $\Gamma_{\tilde{w}_0}^w$ is a p -dimensional face of $\Gamma_{\tilde{w}_0}$ and so it is a convex polyhedral cone. Let \tilde{w}'_0 be another reduced decomposition of w_0 which is not supposed adapted to w . Then, by [1, 3.3], $\Gamma_{\tilde{w}'_0}^w$ is the image of $\Gamma_{\tilde{w}_0}^w$ by a continuous piecewise linear map l . To be more

precise, there exists a finite set of (convex) cones \mathcal{C}_i in \mathbb{R}^{n+N} such that $\bigcup \mathcal{C}_i = \mathbb{R}_{\geq 0}^{n+N}$ and such that l is linear on $\Gamma_{\tilde{w}_0}^w \cap \mathcal{C}_i$. Hence, we have the lemma. \diamond

We still denote by v_λ a highest weight vector of the classical Weyl module $V(\lambda)$. The algebra $A_q[G/U]$ specializes for $q = 1$ onto the algebra $\mathbb{C}[G/U] = \bigoplus_\lambda V(\lambda)^* \otimes v_\lambda$. The algebra $A_q[G/U]/I_{A,w}$ specializes for $q = 1$ onto the quotient algebra $\mathbb{C}[G/U]/I_w = \bigoplus_\lambda V_w(\lambda)^* \otimes v_\lambda$, where $I_w := \bigoplus_{\lambda \in P^+} V_w(\lambda)^\perp \otimes v_\lambda$.

Let \tilde{w}_0 be adapted to w . By the previous lemma, $\Gamma_{\tilde{w}_0}^w$ is generated as a semigroup by a part of the minimal set of generators (λ_i, ψ_i) of $\Gamma_{\tilde{w}_0}$ and with the corresponding relations. This implies :

Theorem. *Choose a reduced decomposition \tilde{w}_0 adapted to w . Then, the graded associated algebra of the quotient filtration of $(\mathcal{F}_m)_{m \in \mathbb{N}}$ on $\mathbb{C}[G/U]/I_w$ is the \mathbb{C} -algebra $\mathbb{C}[\Gamma_{\tilde{w}_0}^w]$ of the semigroup $\Gamma_{\tilde{w}_0}^w$. \diamond*

Remark. Note that this filtration (and hence the corresponding degeneration) depends on the choice of the \mathbb{N} -form e of Lemma 3.2, and of the reduced decomposition \tilde{w}_0 adapted to w . For a general reduced decomposition \tilde{w}'_0 of w_0 , Lemma 3.3 implies that the spectrum of the associated graded algebra is a union of irreducible components which are toric varieties.

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